Evidences of Bolgiano scaling in 3D Rayleigh-Bénard convection

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We present new results from high-resolution high-statistics direct numerical simulations of a tri-dimensional convective cell. We test the fundamental physical picture of the presence of both a Bolgiano-like and a Kolmogorov-like regime. We find that the dimensional predictions for these two distinct regimes (characterized respectively by an active and passive role of the temperature field) are consistent with our measurements.

The dimensional theory of homogeneous and isotropic turbulence has been definitely settled long ago by the work of A. N. Kolmogorov [1]. On the other hand, it is still missing a clear theoretical picture for the strong fluctuations in the energy dissipation field that lead to intermittency effects (i.e. non gaussian behaviour of probability distribution functions). Phenomenological theories have been proposed [1] but no systematic theory for computing experimentally measured numbers has been successful so far. The situation of "non-ideal" turbulence is even more controversial, already at the level of dimensional expectations. A typical realization, the one we will address in this paper, is the tri-dimensional (3D) Rayleigh-Bénard cell, described, in the Boussinesq approximation [2], by the following set of equations:

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} = -\nabla p + \nu\nabla^2\boldsymbol{v} + \alpha g T\hat{\boldsymbol{z}} \tag{1}$$

$$\partial_t T + (\boldsymbol{v}\cdot\nabla)\,T = \chi\nabla^2 T \tag{2}$$

with isothermal boundary conditions on the upper and lower planes of a cell of height H: $T(z=0) = +\Delta T/2$ and $T(z=H) = -\Delta T/2$. As usual, $\boldsymbol{v}(\boldsymbol{x},t)$ is the velocity field and $T(\boldsymbol{x},t)$ the temperature field. Kinematic viscosity and thermal conductivity are respectively ν and χ, while the thermal expansion coefficient is α and gravity acceleration is g. In the following we will mainly focus on the longitudinal structure functions of v and T: $S_p(r) = \langle\, [(\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{x}))\cdot\hat{\boldsymbol{r}}]^p\,\rangle$ and $T_p(r) = \langle\, [T(\boldsymbol{x}+\boldsymbol{r}) - T(\boldsymbol{x})]^p\,\rangle$.

In this work we present some tests of the predictions for the structure functions defined above that can be derived in the scenario proposed years ago by Bolgiano [3] to describe convective turbulence.

Despite much research on the subject, sound evidence of the validity of the Bolgiano scenario and the recovery of Kolmogorov scaling at small scales is still missing. Good quality confirmation of the dimensional Bolgiano scenario was recently shown in a two dimensional numerical simulations [11]. This result cannot be directly related to the 3D case because of the strong differences in the properties of the velocity field in 2D. Statistical properties of the velocity field were recently used with even simpler models (shell models for turbulence) [12]. This approach is even less probing since these models were built precisely in order to implement Bolgiano scaling. Finally, a recent experiment even questioned the behaviour the the Bolgiano length inside a convective cell [10]. Of course from an experimental point of view it can be difficult, if not impossible at all, to measure all relevant quantities and it can be even harder to have access to information at several (not just a few) positions inside the volume. The results presented in this letter show good consistency with the idea of a Bolgiano regime (i.e. a range of length scales where temperature driven buoyancy effects are dominant) but, because of the still limited resolution of "state of the art" numerical simulations, we will have to resort to somehow indirect tests.

This letter is organized as follows: a brief review of phenomenological expectations, details of our numerical simulations, data analysis and then concluding remarks.

Starting from equations (1) and (2), if one uses dimensional analysis and assumes homogeneous scaling for velocity and temperature differences (inside the inertial range), one ends up with two distinct scaling regimes. At small scales (Kolmogorov-like scenario), $r \ll L_B$,

$$\delta v(r) \sim \varepsilon^{1/3} r^{1/3} \tag{3}$$

$$\delta T(r) \sim N^{1/2}\varepsilon^{-1/6} r^{1/3} \tag{4}$$

while at large scales (Bolgiano-like scenario), $r \gg L_B$,

$$\delta v(r) \sim (\alpha g)^{2/5}\, N^{1/5} r^{3/5} \tag{5}$$

$$\delta T(r) \sim (\alpha g)^{-1/5}\, N^{2/5} r^{1/5} \tag{6}$$

The Bolgiano length, L_B, is an estimate of the distance at which the dissipative and buoyancy terms on the right hand side of equation (1) balance, valid under the assumption of a scaling behaviour. In the following we will use the z dependent version of $L_B(z)$ introduced in [4] that uses averages of ε and N defined at a given height z, $\varepsilon(z) = \nu/2\langle \sum_{ij}(\partial_i v_j)^2\rangle_z$ and $N(z) = \chi/2\langle \sum_i (\partial_i T)^2\rangle_z$:

$$L_B(z) = \frac{\varepsilon(z)^{5/4}}{N(z)^{3/4}(\alpha g)^{3/2}} \tag{7}$$

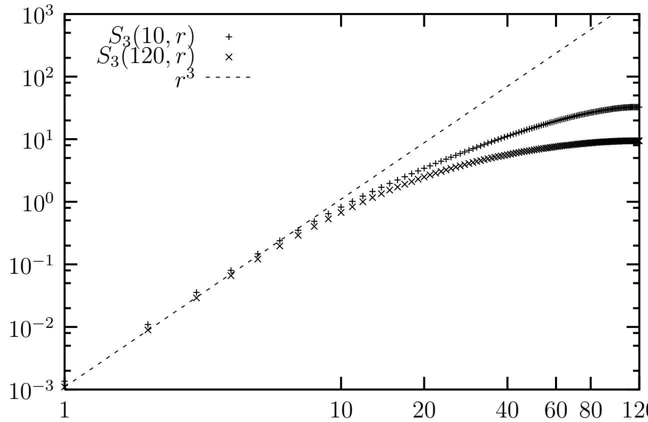

FIG. 1. Log-log plot of the structure function $S_3(r)$, defined in the text, measured close to the end plates (+) and at the center of the cell (×). The horizontal scale is in grid points, while the vertical scale is in arbitrary units.

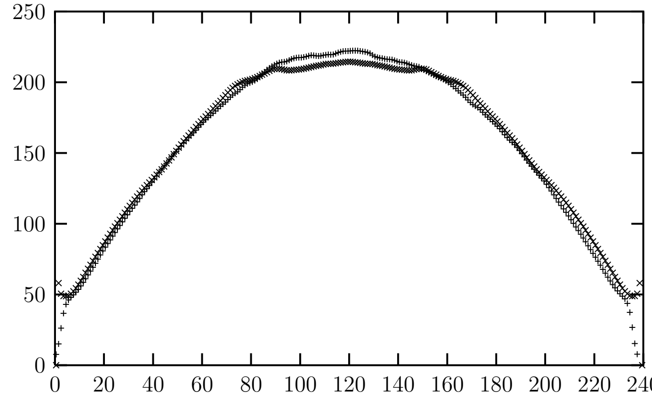

FIG. 2. Plot of the z-dependent Bolgiano length as a function of the distance from the bottom wall, z, as given by the usual definition of eq. 7 ($L_B(z)$, +) and as given by the procedure described in the text, following eq. 8 ($\alpha \cdot \tilde{L}_B(z)$, ×). All dimensions are in grid points and $\alpha = 3.1$.

Our analysis has been performed on data coming from Direct Numerical Simulations (DNS) employing a by now standard Lattice Boltzmann scheme [5], on a massively parallel computer [6]. The resolution of the numerical simulation was 240^3 and the Rayleigh number ($Ra = \alpha g \Delta T H^3/(\nu\chi)$) was approximately $3.5 \cdot 10^7$. The Prandtl number was equal to unity. We performed a stationary simulation extending over approximately 500 recirculation times and stored nearly 400 independent configurations with full information on all velocity components and the temperature field. Boundary conditions were periodic in the x and y directions (in order to maintain homogeneity on horizontal planes) and isothermal at the top and bottom planes of the cell ($z = 0$ and $z = H$).

Free slip boundary conditions (i.e. satisfying only the incompenetrability constraint) were used on the top/bottom planes for the velocity field. This choice was made in an attempt to reduce the effects of a viscous boundary layer close to the horizontal walls. Indeed, as will be clear from the following, thermal effects are dominant near the isothermal walls, so using no-slip boundary conditions might produce effects on the velocity statistics interfering with the ones coming from pure buoyancy. More details on this point will be given later on.

The most direct way to test the dimensional validity of the Bolgiano picture would be to measure structure functions and to check whether they scale with exponents close (apart from intermittency corrections) to the ones predicted by the sets of equations (3-4) and (5-6).

Unfortunately this cannot be done directly because of the limited resolution of our DNS.

Fig. 1 substantiates this comment, by plotting the velocity structure function of order 3, $S_3(z, r) = \langle\, (v(x + r, z) - v(x, z))^3 \,\rangle$ at two different position: one very close to the wall ($z = 10$), and the other at the center of the cell ($z = 120 = H/2$).

As it can be seen from Fig. 1, no evident scaling range can be detected, even if a steepening of $S_3(r)$ is clearly detected as one comes close to the walls.

Given this state of affair, in order to test the consistency with the Bolgiano picture we performed two distinct, even if somehow less direct, tests.
The first test consists in checking that the Bolgiano length actually keeps track of the scales at which the buoyancy term balances the dissipative term in equation (1).

First, we measure $L_B(z)$ in terms of $\varepsilon(z)$ and $N(z)$ and plot our results in Figure 2. From a first look at the behaviour of $L_B(z)$ we learn that Bolgiano effects should be measurable, if at all, near to the isothermal walls (where $L_B(z)$ is of the order of 10^1).

Close to the center of the cell, L_B is of the order of 10^2 and Kolmogorov like behaviour is expected to be measurable at almost all scales.

We then measure, directly and independently from the previous quantity, the scale at which dissipation and buoyancy effects balance, i.e. we look for the scale $\tilde{L}_B(z)$ for which:

$$\varepsilon(z) \sim \alpha g \langle \delta v(\tilde{L}_B) \delta T(\tilde{L}_B) \rangle_z \tag{8}$$

We provisionally consider $\tilde{L}_B$ a modified definition of the Bolgiano length. In Fig. 2 we plot $L_B(z)$ and $\alpha \tilde{L}_B(z)$ with a constant which was tuned to be $\alpha \sim 3.1$.

As it can be seen, the two definitions yield the same behaviour apart from the multiplicative factor, α. The reason for the multiplicative factor (of order unity) is due to the fact that the two definitions are dimensional estimates so they can miss a numerical prefactor. Considering this, the fact that the two definition behave in the same way after rescaling has to be considered, in our opinion, an excellent agreement.

Here we want to underline two points which, we believe, add relevance to this finding. First of all the

two definitions are of course linked but definitely different. The "traditional" definition of L_B (see also [4]), as from definition (7), comes from supposing that the two scaling laws in equations (3) and (5) merge at L_B, hence comes from solving the equation $\varepsilon(z)^{1/3} L_B(z)^{1/3} = (\alpha g)^{2/5} N(z)^{1/5} L_B(z)^{3/5}$. The second definition, $\tilde{L}_B(z)$ comes instead from a *direct* measurement of the strength of the dissipative and forcing term in equation (1).

The second important point consist on the fact that our cell is *not* homogeneous; this adds strength to the equality between the two different definitions of $L_B(z)$.

As a consequence of this test we can claim that the Bolgiano scenario and the expected power law behaviour are consistent with the value measured for the terms appearing on the right hand side of equation (1). Furthermore we fully confirm (with higher accuracy) our former results for L_B [4]. We like to underline that extracting the behaviour of the Bolgiano length using measured quantities that are not those appearing in its definition could introduce large errors. For example in [10] a Bolgiano length was extrapolated as the scale at which there is a change of slope in a particular structure function. In using such a procedure one is dominated by strong finite size effects present on the structure functions.

We now proceed to our second test. Under the hypothesis of validity of Bolgiano scaling, and if enough resolution were available, one would expect to see power law behaviour in the inertial range in Fig. 1 with two distinct slopes 1 and 9/5, corresponding respectively to Kolmogorov and Bolgiano scaling. With available resolution, we are not able to detect a clear power law behaviour from Fig. 1, although we see a clear steepening of the structure function as we come close to the wall. What we are going to do in the following is to try to quantify as well as possible this change of slope.

We adopt the following procedure. We focus on the plot of $S_6(r)$ versus $S_3(r)$ and apply Extended Self Similarity (ESS, see [7]) in order to detect a trustable plateau in the local slopes. We found this plateau to correspond roughly to distances in the interval $\mathcal{I}^v = [25, 40]$ for the velocity structure function and $\mathcal{I}^T = [15, 30]$ for the structure functions of the temperature field. We then define two other interval slightly shifted to the left $\mathcal{I}^v_- = [20, 35], \mathcal{I}^T_- = [10, 25]$ and to the right $\mathcal{I}^v_+ = [30, 45]$, $\mathcal{I}^T_+ = [20, 40]$. These interval were of course shifted by a reasonable amount, i.e. were still possible highest or lowest estimate for the same plateau.

We finally perform a power law fit to extract a scaling exponent on the structure functions for the velocity and for the temperature in the three intervals defined before.

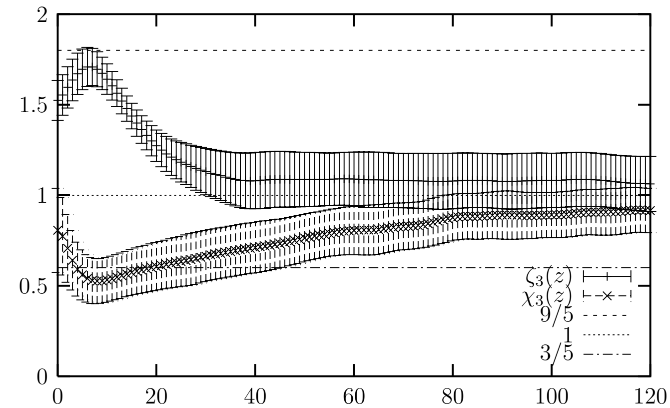

FIG. 3. Behaviour of the exponents $\zeta_3(z)$, $\chi_3(z)$ as a function of z.

We define the scaling exponents for the structure functions of interest as follows:

$$\langle \delta v(z,r)^p \rangle \sim r^{\zeta_p(z)} \tag{9}$$

$$\langle \delta T(z,r)^p \rangle \sim r^{\chi_p}(z) \tag{10}$$

$$\langle \left(\delta v(z,r) \delta T(z,r)^2 \right)^{\frac{p}{3}} \rangle \sim r^{\rho_p(z)} \tag{11}$$

Our fits provide a central value for the exponents and two, respectively, higher and lower estimates (corresponding to the shifted intervals). The procedure adopted in estimating the errors reflects the fact that the largest source of systematic error is connected with the choice of the fitting ranges and not with statistical accuracy.

In the Kolmogorov regime we expect the exponents of equations (9-11) to take the following dimensional values: $\zeta_p = p/3$, $\chi_p = p/3$ and $\rho_p = p/3$. In the Bolgiano regime, on the other hand, we expect the following dimensional values: $\zeta_p = 3/5p$, $\chi_p = p/5$ and $\rho_p = p/3$.

In Figure 3 we plot the behaviour of the measured central exponents and their errors as a function of z for $\zeta_3(z)$ and $\chi_3(z)$. The behaviour is qualitatively and quantitatively consistent with the expected scaling exponents: we observe a smooth transition from a Bolgiano dominated regime (near to the wall, small z) to a Kolmogorov regime (approaching the center of the cell).

In Figure 3 we have not shown the behaviour of $\rho_3(z)$ as it is consistent (within error bars) with the constant value 1 and plotting it on the same figure would have made it unreadable.

Here we, once again, like to underline the consistency of our findings. From the theoretical picture we expect to see the crossover to the Kolmogorov scaling when $L_B(z) \leq H$. This is indeed the case as $L_B(z) \leq H$ for $z \leq 40$ and indeed we see $\zeta_3(z) \sim 1$ in the same range of z.

Another interesting question concerns the "real" statistics of the Bolgiano regime. Very recently interest is growing along this line of research because of the desire to understand the differences of the statistical properties of

an active with respect to a passive scalar [11,12]. In this work we focused on the gross features, i.e. on the dimensional behaviour. If we try to look at intermittency, by means of ESS we find a strong increase of intermittency for the velocity field approaching the isothermal walls.

Unfortunately we believe that with our simulation we are not in a position to make any definite statements about intermittency in the Bolgiano dominated regime.

Indeed a study of intermittency in the Bolgiano regime would involves positions nearby the isothermal walls (as only there the Bolgiano length scale is small enough to have an inertial range largely dominated by buoyancy effects). Recently it was found that intermittency increases in the velocity structure functions inside a viscous boundary layer [9]. In an attempt to reduce the viscous boundary layer we decided to apply free-slip velocity boundary conditions to the isothermal walls free slip for the velocity. Unfortunately our idea cannot be successful because of two reasons. First, the free-slip boundary conditions does not completely suppress the boundary layer. A boundary layer thickness can be defined through an extrapolation of $\langle v_z(x)^2 \rangle$, and in our case turned out to be of the order of roughly 15 grid spacings. Second, it was recently realized that mechanical turbulence in a free-slip channel also presents an enhancement of intermittency near to the walls [8].

A procedure to disentangle buoyancy and planar effect is clearly needed to make any definite statement on intermittency in the Bolgiano regime. Otherwise one cannot decide whether a change of intermittency is related to Bolgiano dynamics or to the increase occurring nearby boundary layers.

In order to clarify this point it would be important to perform a simulation of a Rayleigh-Bénard-like system with periodic boundary condition in all directions (i.e. an homogeneous Rayleigh-Bénard cell). We suggest to extend to 3D the study made in 2 dimensions in [11].

In order to do that one could write temperature field as the sum of a linear profile plus a fluctuating part, $T(x,y,z) = T_{\text{lin}}(z) + T'(x,y,z)$, with $T_{\text{lin}} = \Delta T/2 \cdot (1 - 2z/H)$ and obtain:

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} = -\nabla p + \nu \nabla^2 \boldsymbol{v} + \alpha g T \hat{\boldsymbol{z}} \tag{12}$$

$$\partial_t T' + (\boldsymbol{v} \cdot \nabla)\, T' = \chi \nabla^2 T' - \frac{\Delta T}{H} v_z \tag{13}$$

Choosing the parameters in order to have a Bolgiano length as small as possible, one would benefit of a wide range of scales were to study the buoyancy dominated flow. Further advantage of homogeneity would be the natural increase of statistics and also the applicability of tools like $SO(3)$ decomposition, to disentangle anisotropic terms [13,14]. A study of this kind is in progress.

Concluding we have performed a number of basic tests in order to validate the scenario of Bolgiano-Kolmogorov scaling in a convective cell, within the limitations but also advantage of z non homogeneity of our cell. We were able to confirm the transition between the two expected scenarios.

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